5.5 THE DEFINITE INTEGRAL

$$\int_a^b f(x)\,dx$$

5.5.1 Definition:

$$\int_a^b f(x)\,dx$$

- The **definite integral** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.

5.5.2 Theorem:

- If a function f is ***continuous*** on an interval [a ,b], then f is ***integrable*** on [a ,b], and the net signed ***area*** A between the graph of f and the interval [a ,b] is

$$\boxed{A = \int_a^b f(x)\,dx}$$

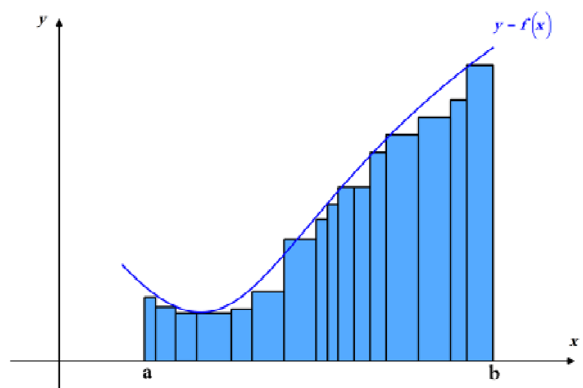

Figure 5.5.2

$$A = \lim_{max\,\Delta x_k \to 0} \sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k$$

$$\boxed{A = \int_a^b f(x)\,dx}$$

5.5.3 Condition of integration:

- A function f is said to be *integrable* on a finite *closed* interval [a ,b] if the limit exists:

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k$$

5.5.4 How to Evaluate:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

✍ Example (1):

- Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2\,dx$ (b) $\int_{-1}^{2} (x+2)\,dx$ (c) $\int_0^1 \sqrt{1-x^2}\,dx$

✎ Solution:

(a) $\int_1^4 2\,dx$

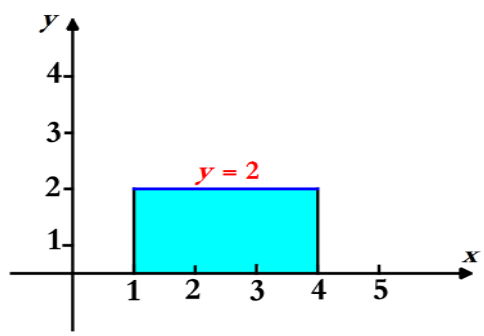

Figure 5.5.4a

- The graph of the integrand is the horizontal line $y = 2$, so the region is a *rectangle* of width 2 extending over the interval from 1 to 4 (*Figure 5.5.4a*). Thus,

$$\int_{1}^{4} 2\,dx = \text{(area of rectangle)}$$

$$= (4 - 1)2 = (3)2 = \boxed{6}$$

(b) $\int_{-1}^{2} (x + 2)\,dx$

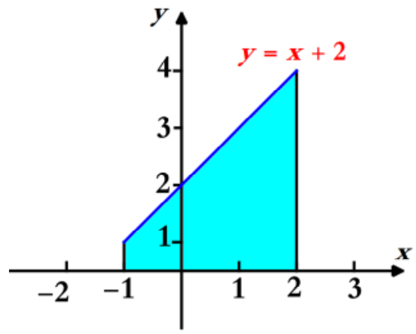

Figure 5.5.4*b*

✸ **The graph of the integrand is the line** $y = x + 2$**, so the region is a** ***trapezoid*** **whose altitude extends from** $x = -1$ **to** $x = 2$ **(*Figure 5.5.4b*). Thus,**

$$\int_{-1}^{2} (x + 2)\,dx = \text{(area of trapezoid)}$$

$$= \frac{1}{2}(1 + 4)(2 - (-1))$$

★ ***Remember that :***

★ *Area of a trapezoid* $A = \frac{1}{2}(b_1 + b_2)h$

★ b_1 *and* b_2 *: lengths of parallel sides*

★ *h: distance between parallel sides or altitude*

$$= \frac{1}{2}(5)(3) = \boxed{\frac{15}{2}}$$

(c) $\int_0^1 \sqrt{1 - x^2}\, dx$

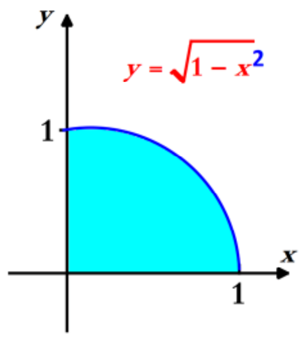

Figure 5.5.4c

- **The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin , so the region is the *right quarter-circle* extending from $x = 0$ to $x = 1$ (*Figure 5.5.4c*). Thus,**

$$\int_0^1 \sqrt{1 - x^2}\, dx = (\text{area of quarter-circle})$$

$$= \frac{1}{4}\pi(1)^2 = \boxed{\frac{\pi}{4}}$$

★ ***Remember that :***

★ *Area of a circle* $A = \pi r^2$

★ ***r: radius of circle***

5.5.3 Properties of the Definite Integral:

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

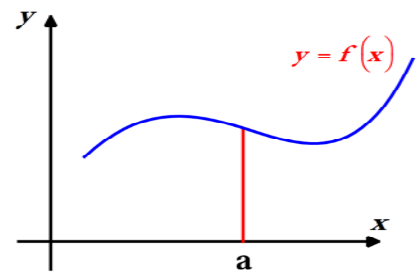

Figure 5.5.6

(b) If f is integrable on [a ,b] , then we define.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

✍ *Example (2):*

(a) $\int_1^1 x^2\,dx = \boxed{0}$

★ ***Remember that :***

★ $\int_a^a f(x)dx = 0$

(b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx$

★ ***Remember that :***

★ $\int_b^a f(x)dx = -\int_a^b f(x)dx$

$= \boxed{-\frac{\pi}{4}}$ **See Example 1 (c)**

5.5.4 Theorem:

- If f and g are *integrable* on [a ,b] and if c is a *constant*, then cf .$f + g$ and $f - g$ are *integrable* on [a ,b]:

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)$

(b) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(c) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

- If f is *integrable* on a *closed* interval containing the three points a, b, and c , then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

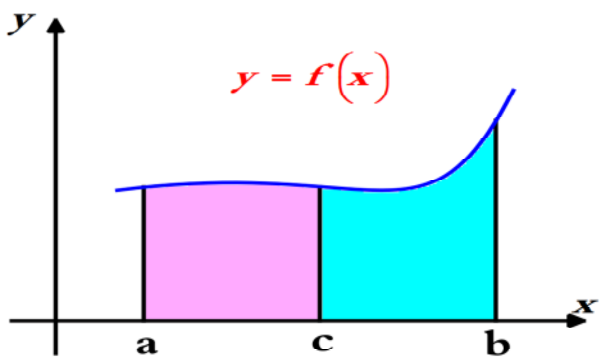

Figure 5.5.7

✍ **Example (3):**

Evaluate:

$$\int_0^1 \left(5 - 3\sqrt{1 - x^2}\right)dx \ .$$

✏ **Solution:**

★ **From parts *(a)* and *(c)* of *Theorem 5.5.4* we can write**

$$\int_0^1 \left(5 - 3\sqrt{1 - x^2}\right)dx = \int_0^1 5\,dx - \int_0^1 3\sqrt{1 - x^2}\,dx$$

$$= \int_0^1 5\,dx - 3\int_0^1 \sqrt{1 - x^2}\,dx$$

✸ **The first integral in this difference can be interpreted as the area of a *rectangle* of length *5* and width *1* , so its value is $\boxed{5}$, and from *Example (1) (c)* the value of the second integral is $\boxed{\pi / 4}$.**

$$\int_0^1 \left(5 - 3\sqrt{1 - x^2}\right)dx = 5 - 3\left(\frac{\pi}{4}\right) = \boxed{5 - \frac{3\pi}{4}}$$

EXERCISE SET 5.5: **(Home Work)**

13(a). Sketch the *region* whose *signed area* is represented by the *definite integral* $\int_0^3 x\,dx$, and evaluate the *integral* using an appropriate formula from *geometry* , where needed.

25. Use *Theorem 5.5.4* and appropriate formula from *geometry* to evaluate the *integral* $\int_{-1}^3 (4 - 5x)\,dx$