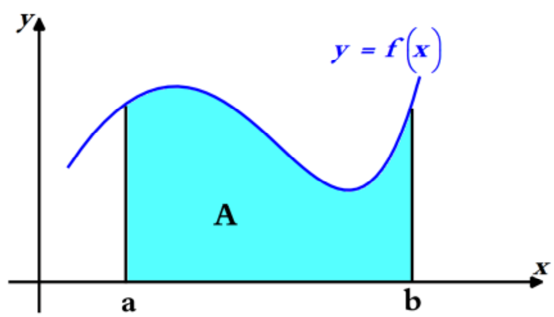
5.6 THE FUNDAMENTAL THEOREM OF CALCULUS



❍ The Fundamental Theorem of Calculus:

★ As in earlier sections, let us begin by assuming that *f* is *nonnegative* and *continuous* on an interval [a ,b], in which case the area *A* under the graph of *f* over the interval [a ,b] is represented by the *definite integral.*

$$A = \int_a^b f(x)\,dx \qquad (1)$$

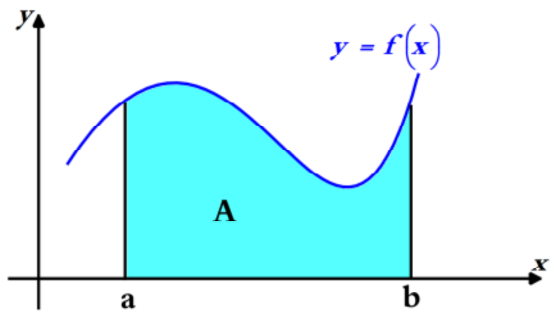


Figure 5.6.1

5.6.1 How to Evaluate:

✸ If *f* is *continuous* on [a ,b] and *F* is any *antiderivative* of *f* on [a ,b], then,

$$\int_a^b f(x)\,dx = \left[F(x) \right]_a^b = F(b) - F(a)$$

★ The *definite integral* can be evaluated by any *antiderivative* of the *integrand* and then *subtracting* the value of this antiderivative at the *lower* limit of *integration* from its value at the *upper* limit of *integration*.

✍ ***Example (1):***

Evaluate:

$$\int_{1}^{2} x\, dx .$$

✏ **Solution:**

✸ **The function** $F(x)=\frac{1}{2}x^2$ **is an** ***antiderivative*** **of** $F(x)=x$ **; thus from (2).**

$$\int_{1}^{2} x\, dx=\left[\frac{1}{2}x^2\right]_1^2=\frac{1}{2}\left[(2)^2-(1)^2\right]$$

$$=\frac{1}{2}(4-1)=\boxed{\frac{3}{2}}$$

✍ ***Example (2):***

Evaluate:

$$\int_{0}^{3}\left(9-x^2\right)dx .$$

✏ **Solution:**

★ *Using the **Fundamental Theorem of Calculus**:*

$$\int_0^3 \left(9 - x^2\right) dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(9(3) - \frac{(3)^3}{3}\right) - 0$$

$$= 27 - 9 = \boxed{18}$$

(a) If f is *integrable* on [a ,b] and $f(x) \geq 0$ for all x in [a ,b], then

$$\int_a^b f(x)\,dx \geq 0$$

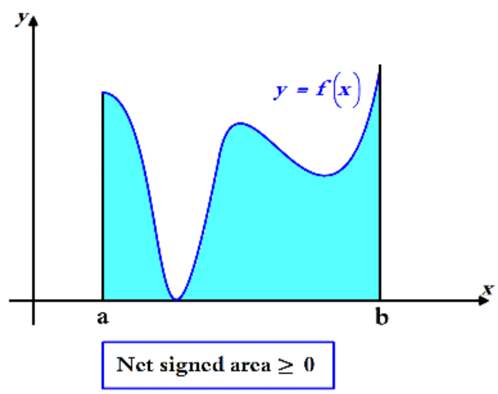


Figure 5.5.8

(b) If f is *integrable* on [a ,b] and $f(x) \leq 0$ for all x in [a ,b], then

$$\int_a^b f(x)\,dx \leq 0$$

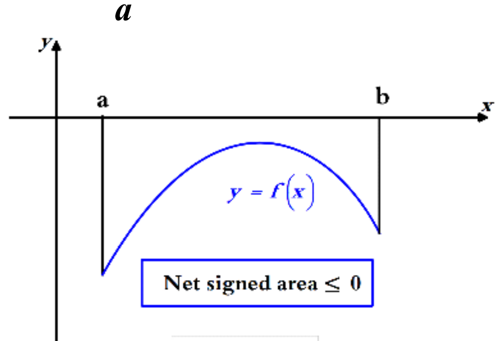


Figure 5.5.9

Example (3):

(a) Find the area under the curve $y = \cos x$ over the interval $\left[0\ ,\ \frac{\pi}{2}\right]$.

(b) Make a conjecture about the value of the integral.

$$\int_0^{\pi} \cos x \, dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

Solution:

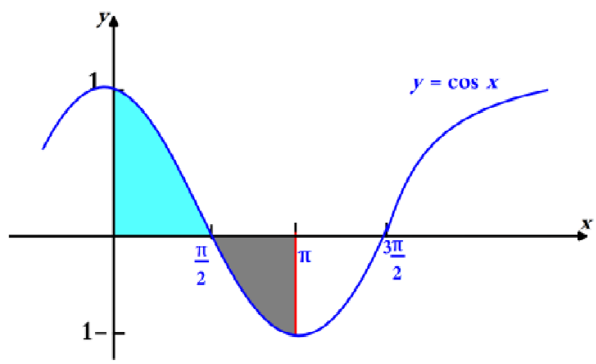


Figure 5.6.4

(a) Since $\cos x \geq 0$ over the interval $[0\ .\ \pi/2]$, the *area A* under the curve is

$$A = \int_0^{\pi/2} \cos x \ dx = [\sin x]_0^{\pi/2}$$

★ ***Remember that:***

$$\star \ \int \cos x \ dx = \sin x + C$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = \boxed{1}$$

(b) The given *integral* can be *interpreted* as the *signed area* between the graph of $y = \cos x$ and the interval $[0 . \pi]$. The graph in *Figure 5.6.4* suggests that over the interval $[0 . \pi]$ the portion of area above the x-axis is the same as the portion of *area* below the *x-axis*, so we conjecture that the *signed area* is **zero** ; this implies that the value of the integral is *zero*. This is *confirmed* by the computations

$$\int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = \boxed{0}$$

✍ Example (4):

$$\int_1^9 \sqrt{x} \, dx = \int_1^9 x^{1/2} \, dx = \frac{2}{3}\left[x^{3/2}\right]_1^9$$

$$= \frac{2}{3}\left[(9)^{3/2} - (1)^{3/2}\right] = \frac{2}{3}(27 - 1) = \boxed{\frac{52}{3}}$$

✍ Example (5):

✸ *Try to solve:*

$$\int_4^9 x^2 \sqrt{x} \, dx$$

✏ **Solution:**

❖ *The Difference Between Definite and Indefinite Integrals:*

✷ Indefinite Integrals:

$$\int f(x)\,dx = [F(x) + C]$$

✷ Definite Integrals:

$$\int_a^b f(x)\,dx = [F(x) + C]_a^b$$

$$= [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

yields,

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

$$= [F(x)]_a^b = F(b) - F(a)$$

Thus, for purposes of evaluating a *definite integral* we can *omit* the *constant of integration.*

Example (6):

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx$$

Solution:

✍ Example (7):

★ $\int_{0}^{\pi/3} sec^2 x \, dx$

✏ Solution:

- **Remember that:**

$$\left| \int sec^2 x \, dx = tan \; x + C \right.$$

$$\frac{\pi}{3} = \frac{\pi}{3} \cdot \frac{180^o}{\pi} = 60^o \;\; . \; tan \; 0 = 0 \; . \; tan \frac{\pi}{3} = \sqrt{3}$$

$$\int_{0}^{\pi/3} sec^2 x \, dx = [tan x]_{0}^{\pi/3}$$

$$= tan\left(\frac{\pi}{3}\right) - tan\,(0) = \sqrt{3} - 0 = \boxed{\sqrt{3}}$$

✍ Example (8):

★ $\int_{0}^{ln3} 5 e^x \, dx$

✏ Solution:

- **Remember that:**

$$\left| \int e^x \, dx = e^x + C \right.$$

$$\int_{0}^{ln3} 5 e^x \, dx = 5 \int_{0}^{ln3} e^x \, dx = 5 [e^x]_{0}^{ln\,3}$$

$$= 5 \left(e^{ln3} - e^0\right) = 5[3 - 1] = \boxed{10}$$

✍ **Example (9):**

★ $\int_{-e}^{-1} \frac{1}{x}\, dx$

✏ Solution:

○ ***Remember that:***

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int_{-e}^{-1} \frac{1}{x}\, dx = \left[\ln|x|\right]_{-e}^{-1}$$

$$= \ln|-1| - \ln|-e| = \ln 1 - \ln e = 0 - 1 = \boxed{-1}$$

✍ **Example (10):**

★ $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx$

✏ Solution:

○ ***Remember that:***

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1} x + C$$

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx = \left[\sin^{-1} x\right]_{-1/2}^{1/2}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \boxed{\frac{\pi}{3}}$$

- If f is *integrable* on a *closed* interval containing the three points a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

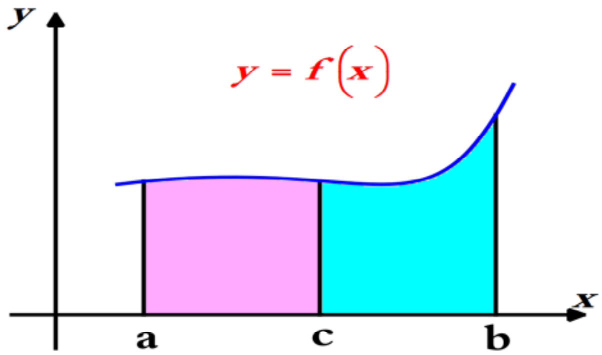


Figure 5.6.5

Example (11):

Evaluate:

$$\int_0^3 f(x)\,dx \quad if$$

$$f(x) = \begin{cases} x^2 & . \quad x < 2 \\ 3x - 2 & . \quad x \geq 2 \end{cases}$$

Solution:

Graph:

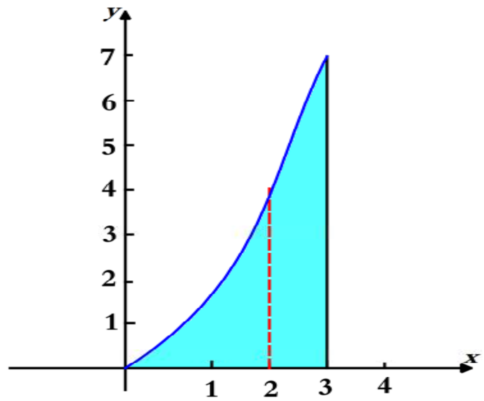


Figure 5.6.5

$$\int_0^3 f(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx$$

$$\int_0^2 x^2\,dx + \int_2^3 (3x - 2)dx$$

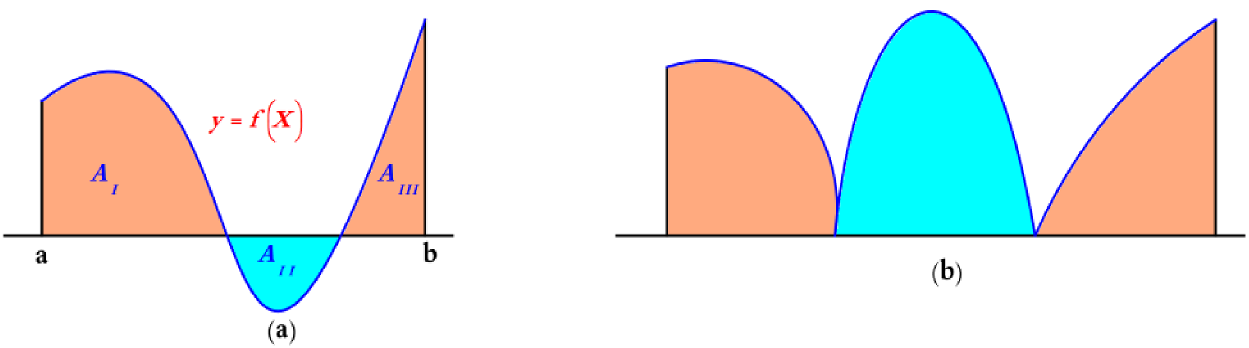
$$= \left[\frac{x^3}{3}\right]_0^2 + \left[3\frac{x^2}{2} - 2x\right]_2^3$$

$$= \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \boxed{\frac{49}{6}}$$

❖ The Total Area:

- If f is a *continuous* function on the interval [a ,b], then we define the *total area* between the curve $y = f(x)$ and the interval [a ,b], to be

$$total\ area = \int_a^b |f(x)|dx$$



$$Total\ area = A_I + A_{II} + A_{III}$$

Figure 5.6.6

- To compute *total area,* being by dividing the interval of integration into *subintervals* on which $f(x)$ does *not change* sign. On the *subintervals* for which $f(x) \geq 0$ replace $|f(x)|$ by $f(x)$, and on the *subintervals* for which $f(x) \leq 0$ replace $|f(x)|$ by $-f(x)$. *Adding* the resulting intervals then yields the *total area.*

✍ ***Example (12):***

✸ **Find the total area between the curve y $y = 1 - x^2$ and the x-axis over the interval [0,2] (*Figure 5.6.7*).**

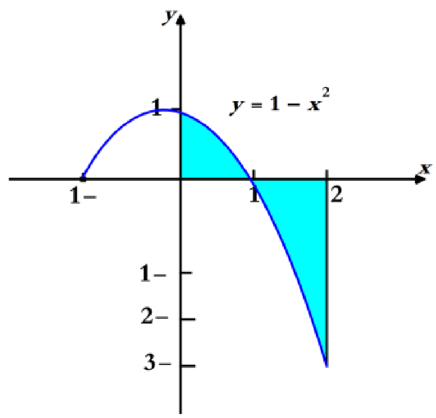


Figure 5.6.7

✐ ***Solution:***

* *The area A is given by*

$$A = \int_a^b |f(x)|\,dx = \int_0^2 |1 - x^2|\,dx$$

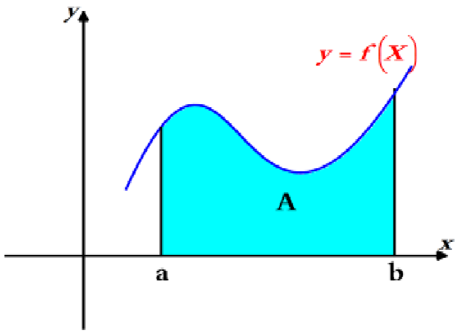
$$= \int_0^1 \left|(1 - x^2)\right| dx + \int_1^2 -(1 - x^2)\,dx$$

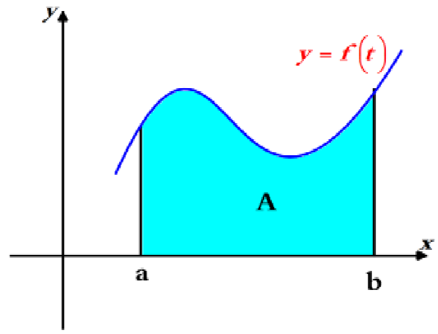
$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = \boxed{2}$$

Dummy Variables:

- The *area* under the graph of the curve $y = f(x)$ over an interval [a ,b] on the x-axis is the *same* as the *area* under the graph of the curve $y = f(t)$ over the interval [a ,b] on the taxis (*Figure 5.6.8*).





$$A = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Figure 5.6.8

- Because the variable of integration in a *definite integral* plays no role in the end result, it is often referred to as a *dummy variable*. In summary.

> ▪ Whenever you find it convenient to *change* the *letter* used for the *variable* of *integration* in a *definite integral*, you can do so without *changing* the value of the integral.

Example (13):

Evaluate:

$$\int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta$$

Solution:

Example (14):

Evaluate:

$$\int_{0}^{\pi/3} \left(2t - \sec t \tan t\right) dt$$

Solution:

The Mean-Value Theorem for Integrals:

- Let f be a *continuous nonnegative* function on $[a, b]$, and let m and M be the *minimum* and *maximum* values of $f(x)$ on this interval. Consider the *rectangles* of heights m and M over the interval $[a, b]$ (*Figure 5.6.9*).

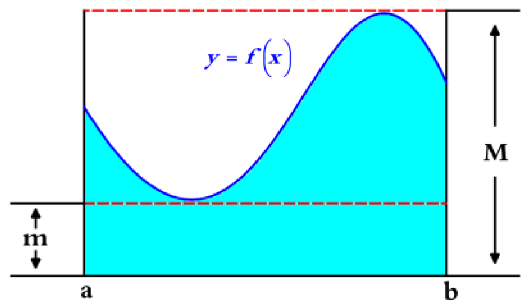


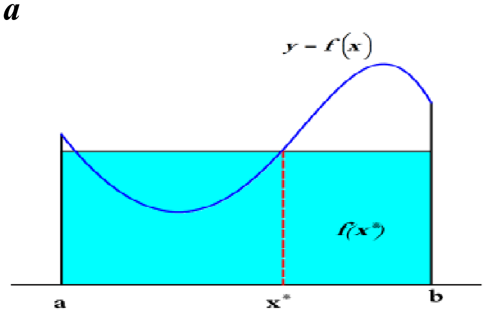
Figure 5.6.9

- It is clear *geometrically* from this figure that the *area*

$$A = \int_a^b f(x)\,dx$$

It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f(x^*)$ between m and M whose area is precisely A; that is,

$$\int_a^b f(x)\,dx = f(x^*)(b - a)$$



The area of the shaded rectangle is equal to the area of the shaded region in figure 5.6.9

Figure 5.6.10

✍ **Example (15):**

✸ **Since $f(x) = x^2$ is *continuous* on the interval [1,4] , the *Mean - Value Theorem for Integrals guarantees* that there is a point x^* in [1,4] such that.**

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2\,(4-1) = 3(x^*)^2$$

✸ **But**

$$\int_1^4 x^2\,dx = \frac{1}{3}\left[x^3\right]_1^4 = \frac{1}{3}\left[(4)^3 - (1)^3\right] = \frac{1}{3}(64-1) = \boxed{21}$$

so that,

$$3(x^*)^2 = 21 \quad or \quad (x^*)^2 = 7 \quad or \quad x^* = \pm\sqrt{7}$$

✸ **Thus, $\boxed{x^* = \sqrt{7} \approx 2.65}$ is the point in the interval [1,4] whose existence is *guaranteed* by the *Mean-Value Theorem for Integrals.***

✍ **Example (16):**

✸ **Since $f(x) = 2x\sqrt{x}$ is *continuous* on the interval [4,9] , the *Mean -Value Theorem for Integrals guarantees* that there is a point x^* in [4,9].**

✍ **Solution:**

General Area at any point (x):

- In *Section 5.1* we suggested if f is *continuous* and *nonnegative* on [a ,b], and if $A(x)$ is the *area* under the graph of $y = f(x)$ over the interval (*Figure 5.6.2*), then $A(x)$ can be expressed as the *definite integral* .

$$A(x)=\int_{a}^{x} f(t)\,dt$$

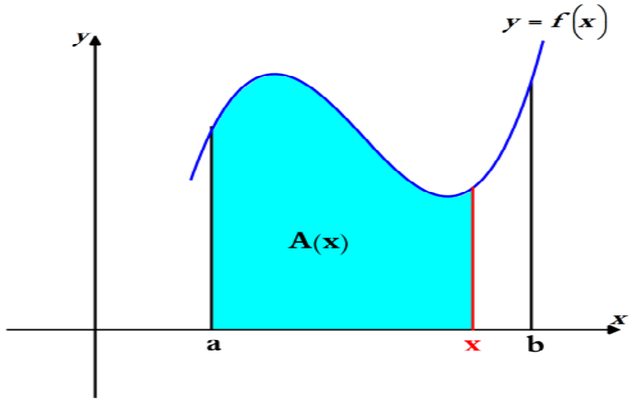


Figure 5.6.2

(Where we have used *t* rather than *x* as the variable of integration to avoid confusion with the *x* that appears as the *upper* limit of *integration*).

✍ ***Example (17):***

- **Try To Find the value of A(x) for area under: f(x) = e^x – cos(x) at interval [0, $\frac{\pi}{2}$].**

✸ Thus, the relationship $A'(x) = f(x)$ can be expressed as

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

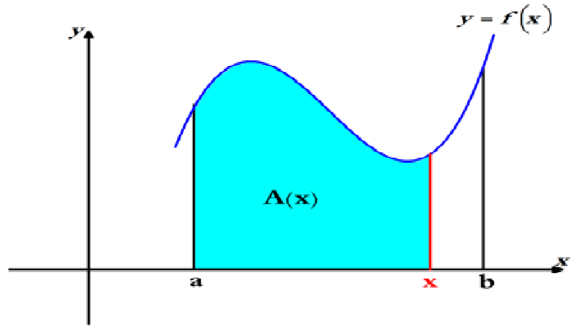


Figure 5.6.2

> ***If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.***

Example (18):

✸ ***Evaluate:***

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

✎ Solution:

✸ The *integrand* is a *continuous* function, so from **(11)**

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = \boxed{x^3}$$

✷ Alternatively, evaluating the *integral* and then *differentiating* yields

$$\int_{1}^{x} t^{3}\,dt = \frac{1}{4}\left[t^{4}\right]_{t=1}^{x} = \frac{1}{4}\left(x^{4} - 1\right)$$

$$\frac{d}{d\,x}\left[\frac{1}{4}\left(x^{4}-1\right)\right] = \boxed{x^{3}}$$

✍ **Example (19):**

✍ **Evaluate:**

$$\frac{d}{dx}\left[\int_{1}^{x} \sin\left(t^{2}\right)dt\right]$$

✍ **Example (20):**

✷ **Evaluate:**

$$\frac{d}{dx}\left[\int_{0}^{x} e^{\sqrt{t}}\,dt\right]$$