# 5.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

$$\int_a^b f \quad g \quad x \quad g' \quad x \quad dx$$

## ❖ Two Methods for Making Substitutions in Definite Integrals :

★ Recall from Section 5.3 that indefinite integrals of the form

$$\int f(g(x))\,g'(x)\,dx$$

- Can sometimes be evaluated by making the u-substitution

$$\boxed{u = g(x)} \quad . \quad \boxed{du = g'(x)\,dx} \qquad (1)$$

- Which converts the integral to the form

$$\int f(u)\,du$$

★ To apply this method to a definite integral of the form

$$\int_a^b f(g(x))\,g'(x)\,dx$$

- There are two ways of doing this:

### ❍ Method 1:

✸ First evaluate the indefinite integral

$$\int f(g(x))\,g'(x)\,dx$$

by substitution , and then use the relationship

$$\int_a^b f(g(x))\,g'(x)\,dx = \left[\int f(g(x))\,g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

## Example (1) :

★ *Use the two methods above to evaluate* $\int_0^2 x\left(x^2+1\right)^3\,dx$ .

## Solution:

$$\int_0^2 x\left(x^2+1\right)^3\,dx$$

### Method (1) :

✸ *If we let*

$\boxed{u = x^2 + 1}$ *so that* $du = 2x\,dx \quad \left(\text{or } \frac{1}{2}\,du = x\,dx\right)$

*Then we obtain*

$$\int x\left(x^2+1\right)^3 dx = \frac{1}{2}\int u^3\,du$$

$$= \frac{1}{2}\left(\frac{u^4}{4}\right) + C = \frac{1}{8}\left(x^2+1\right)^4 + C$$

*Thus ,*

$$\int_0^2 x\left(x^2+1\right)^3 dx = \left[\int x\left(x^2+1\right)^3 dx\right]_{x=0}^{2}$$

$$= \left[\frac{1}{8}\left(x^2+1\right)^4\right]_0^2 = \frac{1}{8}\left[\left(2^2+1\right)^4 - \left(0^2+1\right)^4\right]$$

$$= \frac{1}{8}\left(625 - 1\right) = \frac{624}{8} = \boxed{78}$$

## ❍ Method 2:

✸ *Make the substitution (1) directly in the definite integral , and then use the relationship* $\boxed{u = g\ \ x}$ *to replace the x-limits ,* $x = a$ *and* $x = b$ *, by corresponding u-limits ,* $u = g\ \ a$ *and* $u = g\ \ b$ *.This produces a new definite integral*

$$\int_{g\ \ a}^{g\ \ b} f\ \ u\ \ du$$

*that is expressed entirely in terms of u.*

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$$\int_{0}^{2} x\ \ x^{2} + 1^{\ \ 3}\ \ dx$$

## ✎ Solution:

### ❖ Method (2):

✸ *If we let*

$$\boxed{u = x^{2} + 1} \quad \text{so that} \quad du = 2\,x\,dx \quad \left( or \ \ \frac{1}{2}\ du = x\,dx \right)$$

*Then,*

$$\text{If} \ \ x = 0 \ \ . \ \ u = \ 0^{\ \ 2} + 1 \ = \ 1$$

$$\text{If} \ \ x = 2 \ \ . \ \ u = \ 2^{\ \ 2} + 1 \ = \ 5$$

*Thus ,*

$$\int_0^2 x\left(x^2+1\right)^3 dx = \frac{1}{2}\int_1^5 u^3\, du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_{u=1}^{5} = \frac{1}{8}\left[\left(5\right)^4 - \left(1\right)^4\right]$$

$$= \frac{1}{8}\left(625 - 1\right) = \frac{624}{8} = \boxed{78}$$

- ***Which agrees with the result obtained by Method 1.***

## ✍ Example (2):

**Evaluate**

(a) $\displaystyle\int_{0}^{\pi/8} \sin^5 2x \cos 2x \, dx$  (b) $\displaystyle\int_{2}^{5} (2x-5)(x-3)^9 \, dx$

## ✏ Solution:

(a) $\displaystyle\int_{0}^{\pi/8} \sin^5 2x \cos 2x \, dx = \int_{0}^{\pi/8} (\sin 2x)^5 \cos 2x \, dx$

★ *If we let*

$u = \sin 2x$ so that $du = 2\cos 2x \, dx$

Or $\frac{1}{2} du = \cos 2x \, dx$

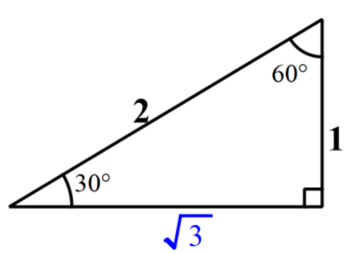
*With this substitution,*

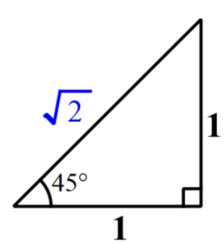
If $x = 0$ . $u = \sin(0) = 0$

If $x = \pi/8$ . $u = \sin(\pi/4) = 1/\sqrt{2}$

★ **Remember that :**

★ $\frac{\pi}{4} = \frac{\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 45^\circ$ . $\sin 0 = 0$ . $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



*Thus ,*

$$\int_{0}^{\pi/8} \sin^5 2x \cos 2x \, dx = \frac{1}{2} \int_{0}^{1/\sqrt{2}} u^5 \, du$$

$$= \frac{1}{2}\left[\frac{u^6}{6}\right]_{u=0}^{1/\sqrt{2}} = \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^6 - 0^{\;6}\right]$$

$$= \frac{1}{12}\left(\frac{1}{8}\right) = \boxed{\frac{1}{96}}$$

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(b) $\int_{2}^{5} 2x - 5 \quad x - 3^{\;9} \, dx$

★ *If we let*

$\boxed{u = x - 3}$ *so that* $du = dx$

*This leaves a factor of* $2x - 5$ *unresolved in the integrand.*

*However,*

$$x = u + 3 \;, \quad so \quad 2x - 5 = 2 \; u + 3 \; - 5 = 2u + 1$$

*With this substitution ,*

$$If \;\; x = 2 \;\; . \;\; u = 2 - 3 = -1$$

$$If \;\; x = 5 \;\; . \;\; u = 5 - 3 = 2$$

*Thus ,*

$$\int_{2}^{5} 2x - 5 \quad x - 3^{\;9} \, dx = \int_{-1}^{2} 2u + 1 \; u^9 \, du$$

$$= \int_{-1}^{2} 2u^{10} + u^9 \quad du$$

$$= \left[ \frac{2u^{11}}{11} + \frac{u^{10}}{10} \right]_{u=-1}^{2}$$

$$= \left( \frac{2\ 2^{\ 11}}{11} + \frac{2^{\ 10}}{10} \right) - \left( \frac{2\ -1^{\ 11}}{11} + \frac{-1^{\ 10}}{10} \right)$$

$$= \boxed{\frac{52.233}{110} \approx 474.8}$$

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## ✍ Example (3):

**Evaluate**

(a) $\displaystyle\int_0^{3/4} \frac{dx}{1 - x}$  (b) $\displaystyle\int_0^{in\ 3} e^x\ 1 + e^{\ 1/2}\ dx$

## ✏ Solution:

(a) $\displaystyle\int_0^{3/4} \frac{dx}{1 - x}$

★ *If we let*

$\boxed{u = 1 - x}$ *so that* $du = -dx$

*Or* $-du = dx$

*With this substitution ,*

*If* $x = 0\ .\ u = 1 - 0 = 1$

*If* $x = 3/4\ .\ u = 1 - 3/4 = 1/4$

*Thus ,*

$$\int_0^{3/4} \frac{dx}{1 - x} = \int_1^{1/4} \frac{-1}{u}\, du$$

★ **Remember that :**

$$★ \int \frac{1}{u}\, du = \ln|u| + C$$

$$= -\left[\ln|u|\right]_{u=1}^{1/4} = -\left[\ln\left|\frac{1}{4}\right| - \ln|1|\right]$$

$$= -\left[\ln|1| - \ln|4| - \ln|1|\right] = \boxed{\ln 4}$$

★ **Remember that :**

*If* $a > 0 \;.\; b > 0$ , *then* $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ , $\ln 1 = 0$

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(b) $\displaystyle\int_0^{\ln 3} e^x \left(1 + e^x\right)^{1/2} dx$

★ *If we let*

$\boxed{u = 1 + e^x}$ *so that* $du = e^x\, dx$

*With this substitution,*

*If* $x = 0 \;.\; u = 1 + e^0 = 1 + 1 = 2$

*If* $x = \ln 3 \;.\; u = 1 + e^{\ln 3} = 1 + 3 = 4$

*Thus ,*

$$\int_0^{\ln 3} e^x \left(1 + e^x\right)^{1/2} dx = \int_2^4 u^{1/2}\, du$$

$$= \left[ \frac{u^{3/2}}{3/2} \right]_{u=2}^{4} = \frac{2}{3}\left[ 4^{3/2} - 2^{3/2} \right]$$

$$= \frac{2}{3}\left[ 2^{3} - 2^{3/2} \right] = \boxed{\frac{4}{3}\left(4 - \sqrt{2}\right)}$$

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***Example (4):***

***Try to solve***: $\int_{0}^{1} \frac{x}{\sqrt{4-3x^{4}}} dx$

## Solution:

## EXERCISE SET 5.9: (Home Work)

- 5,11,12,13,16

  Evaluate the definite integral two ways: first by a u-substitution in the definite integral and then by a u-substitution in the corresponding indefinite integral.

5. $\int_{0}^{1} 2x+1\ ^{3}\, dx$

11. $\int_{0}^{\pi/2} 4 \sin\ x/2\ \, dx$

12. $\int_{0}^{\pi/6} 2 \cos 2x\, dx$

13. $\int_{-2}^{-1} \frac{x}{x+2\ ^{3}}\, dx$

16. $\int_{0}^{\ln 5} e^{x}\ 3 - 4e^{x}\ \, dx$

32,37,38,46,47 Evaluate the integrals by any method.

32. $\int_{1}^{2} \sqrt{5x-1}\, dx$

37. $\int_{0}^{\pi/4} 4 \sin x \cos x\, dx$

38. $\int_{0}^{\pi/4} \sqrt{\tan x}\, \sec^{2} x\, dx$

46. $\int_{1}^{\sqrt{2}} x e^{-x^{2}}\, dx$