# *Lectures of Calculus II*

## *MATH106*

## *Course Description:*

- **Chapter 5: Integration**
- **Chapter 7: Principles of Integral Evaluation**
- **Chapter 6: Applications of the Definite Integral in Geometry, Science, and Engineering**
- **Chapter 9: Infinite Series**

## 5.2 THE INDEFINITE INTEGRAL:

- **Antiderivatives:**

### 5.2.1 Definition:

**A function F is called an antiderivative of a function f on a given open interval if F'(x) = f (x) for all x in the interval.**

- ***For example:*** **the function** $F(x) = \frac{1}{3}x^3$ **is an antiderivative of** $f(x) = x^2$.

**However,** $F(x) = \frac{1}{3}x^3$ **is not the only antiderivative of f on this interval. If we add any constant C to** $\frac{1}{3}x^3$**, then the function** $G(x) = \frac{1}{3}x^3 + C$ **is also an antiderivative of f.**

**In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,**

$$\frac{1}{3}x^3 \ , \ \frac{1}{3}x^3 + 2 \ , \ \frac{1}{3}x^3 - 5 \ , \ \frac{1}{3}x^3 - \sqrt{2}$$

**are all antiderivatives of** $f(x) = x^2$.

### *5.2.2 Theorem:*

**If F (x) is any antiderivative of f (x) on an interval, then for any constant C the function F(x) + C is also an antiderivative on that interval. Moreover, each antiderivative of f (x) on the interval can be expressed in the form F(x) + C by choosing the constant C appropriately.**

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## *The Indefinite Integral :*

**The process of finding antiderivatives is called antidifferentiation or integration. Thus, if**

$$\frac{d}{dx}[F(x)] = f(x) \qquad (1)$$

**then integrating (or antidifferentiating) the function f (x) produces an antiderivative of the form F(x) + C. To emphasize this process, Equation (1) is recast using integral notation,**

$$\int f(x).\,dx = F(x) + c \qquad (2)$$

**Where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notation to express the same fact.**

*For example.*

$$\int x^2\,dx = \frac{1}{3}x^3 + C \qquad \text{is equivalent to} \qquad \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$

**Note that if we differentiate an antiderivative of f (x), we obtain f (x) back again. Thus,**

$$\frac{d}{dx}\left[\int f(x).\,dx\right] = f(x) + c \qquad (3)$$

$$\int f(x).\,dx = F(x) + c$$

**The integral of f (x) with respect to x is equal to F (x) plus a constant C.**

**The differential symbol, dx, in the differentiation and antidifferentiation operations.**

$$\frac{d}{dx}[\,]\ and \int [\,]\,dx$$

**serves to identify the independent variable.**

- **Here are some examples of derivative formulas and their equivalent integration formulas:**

| Derivative Formula | Equivalent Integration Formula |
|---|---|
| $\frac{d}{dx}\left[x^3\right]=3x^2$ | $\int 3x^2dx = x^3 + C$ |
| $\frac{d}{dx}\left[\sqrt{x}\right]=\frac{1}{2\sqrt{x}}$ | $\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C$ |
| $\frac{d}{dx}\left[\tan\ t\right]=\sec^2 t$ | $\int \sec^2 t\ dt = \tan\ t + C$ |
| $\frac{d}{du}\left[u^{3/2}\right]=\frac{3}{2}u^{1/2}$ | $\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$ |

- **For simplicity, the dx is sometimes absorbed into the integrand.**

✍ ***For example,***

$$\int 1\ dx \quad \textit{can be written as} \quad \int dx$$

$$\int \frac{1}{x^2}\ dx \quad \textit{can be written as} \quad \int \frac{dx}{x^2}$$

## ❖ *Integration Formulas:*

**Some of the most important basic integration formulas are given in Table 5.2.1.**

*Table 5.2.1*

| Derivative Formula | Equivalent Integration Formula |
|---|---|
| **1.** $\frac{d}{dx}[x] = 1$ | $\int dx = X + C$ |
| **2.** $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$ | $\int x^r \cdot dx = \frac{x^{r+1}}{r+1} + c$ |
| **3.** $\frac{d}{dx}[sin\, x] = cos\, x$ | $\int \cos(x)\, .dx = \sin(x) + c$ |
| **4.** $\frac{d}{dx}[-cosx] = sin\, x$ | $\int \sin(x) \cdot dx = -\cos(x) + C$ |
| **5.** $\frac{d}{dx}[tanx] = sec^2 x$ | $\int Sec^2(x)\, .dx = \tan(x) + c$ |
| **6.** $\frac{d}{dx}[-cotx] = csc^2\, x$ | $\int csc^2(x)\, .dx = -cot(x) + c$ |
| **7.** $\frac{d}{dx}[secx] = sec\, x\, tan\, x$ | $\int Sec(x)\, tan(x)\, .dx = Sec(X) + C$ |
| **8.** $\frac{d}{dx}[-cscx] = csc\, x\, cot\, x$ | $\int csc(x)\, cot(x)\, .dx = -csc(x) + c$ |
| **9.** $\frac{d}{dx}[e^x] = e^x$ | $\int e^x .dx = e^x + c$ |
| **10.** $\frac{d}{dx}\left[\frac{b^x}{Ln(b)}\right] = b^x$ | $\int b^X \cdot dx = \frac{b^X}{ln(b)} + c$ |
| **11.** $\frac{d}{dx}[In\, \lvert x\rvert] = \frac{1}{x}$ | $\int \frac{1}{x} \cdot dx = Ln(x) + C$ |

## Example (1):

**Evaluate the following integrals.**

★ $\int x^2\,dx = \boxed{\dfrac{x^3}{3} + C}$ $\qquad \boxed{r = 2}$

★ **Remember that:**

> ★ $\int x^2\,dx = \dfrac{x^{r+1}}{r+1} + C \quad (r \neq -1)$

★ $\int x^3\,dx = \boxed{\dfrac{x^4}{4} + C}$ $\qquad \boxed{r = 3}$

★ $\int \dfrac{1}{x^5}\,dx = \int x^{-5}\,dx$ $\qquad \boxed{r = -5}$

★ $= \dfrac{x^{-5+1}}{-5+1} + C = \boxed{-\dfrac{1}{4x^4} + C}$

★ $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$ $\qquad \boxed{r = \dfrac{1}{2}}$

★ $= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{2}{3}x^{\frac{3}{2}} + C = \boxed{\dfrac{2}{3}\left(\sqrt{x}\right)^3 + C}$

## Properties of the Indefinite Integral:

### 5.2.3 Theorem:

**(a)** A constant factor can be moved through an integral sign ; that is ,

$$\int c\, f(x)dx = c\ F(x) + C$$

**(b)** An antiderivative of a sum is the sum of the antiderivatives ; that is ,

$$\int \left[ f(x) + g(x) \right] dx = F(x) + G(x) + C$$

**(c)** An antiderivative of a difference is the difference of the antiderivatives ; that is,

$$\int \left[ f(x) - g(x) \right] dx = F(x) - G(x) + C$$

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### Example (2):

**Evaluate:**

a) $\int 4 \cos x \ dx$

b) $\int \left( x + x^2 \right) dx$

### Solution:

a) $\int 4 \cos x\, dx$ , *Since* $F(x) = Sin\ x$ is an antiderivative for $f(x) = \cos x$ *(Table 5.2.1) , we obtain*

$$\int 4 \cos x\, dx = 4 \int \cos x \ dx = \boxed{4\ Sin\ x + C}.$$

★ **Remember that:**

- $\int c\, f(x)dx = c \int f(x)dx = c\, f(x) + C$
- $\int \cos x\ dx = \sin x + C$

b) $\int \left(x + x^2\right) dx$, from Table 5.2.1 we obtain

$$\int \left(x + x^2\right) dx = \int x \; dx + \int x^2 \; dx = \boxed{\frac{x^2}{2} + \frac{x^3}{3} + C}$$

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## Example (3):

**Evaluate:**

$$\int \left(3x^6 - 2x^2 + 7x + 1\right) dx.$$

### Solution:

$$\int \left(3x^6 - 2x^2 + 7x + 1\right) dx.$$

$$= 3 \int x^6 dx - 2 \int x^2 \; dx + 7 \int x \; dx + \int 1 \, dx \; .$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

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## Example (4):

**Evaluate:**

(a) $\int \frac{\cos x}{\sin^2 x} dx$ (b) $\int \frac{t^2 - 2t^4}{t^4} dt$ (c) $\int \frac{x^2}{x^2 + 1} dx$

### Solution:

(a) $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{\sin x} \; \frac{\cos x}{\sin x} dx$

★ **Remember that :**

★ $\frac{1}{\sin x} + \csc x \quad . \quad \frac{\cos x}{\sin x} = \cot x$

$$= \int \csc x \cot x \, dx = \boxed{-\csc x + C}$$

★ **Remember that :**

$$★ \int \csc x \cot x = -\csc x + C$$

(b) $$\int \frac{t^2 - 2t^4}{t^4} dt = \int \left( \frac{1}{t^2} - 2 \right) dt$$

★ **Remember that :**

$$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c} \quad . \quad \frac{x^n}{x^m} = x^{n-m}$$

$$= \int t^{2} - 2 \; dt = \frac{t^{-1}}{-1} - 2t + C$$

★ **Remember that :**

$$★ \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad r \neq -1$$

$$= \boxed{-\frac{1}{t} - 2t + C}$$

(C) $$\int \frac{x^2}{x^{2+1}} dx = \int \frac{x^2 + 1 - 1}{x^{2+1}} dx$$

**Remember that :**

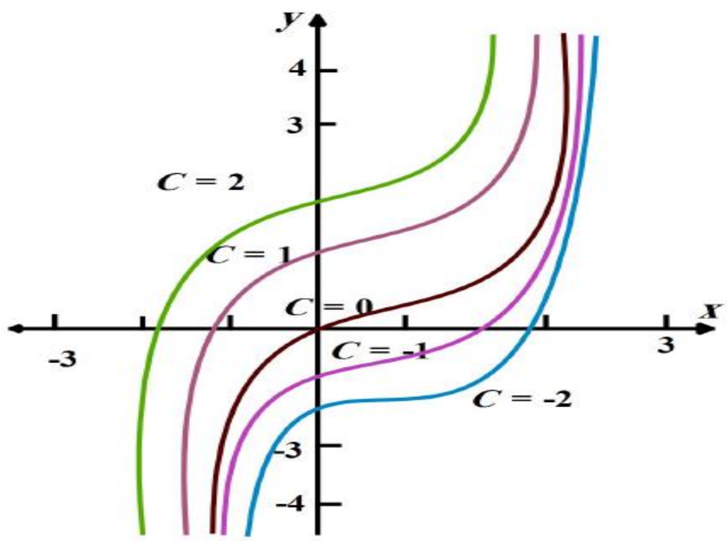
$$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}$$

$$= \int \left( 1 - \frac{1}{x^{2+1}} \right) dx = \boxed{x - \tan^{-1} x + C}$$

## Integral Curves:

$$y = F(x) + C$$

★ *For example,* $y = \frac{1}{3}x^3$ is one integral curve for so $F(x) = x^2$, so all other integral curves have equations of the form $y = \frac{1}{3}x^3$; conversely , the graph of any equation of this form is an integral curve (Figure 5.2.1).




$$y = \frac{1}{3}x^3 + C$$

Figure 5.2.1

## Example (5):

Suppose that a curve $y = f(x)$ in the $y\ x$ - plane has the property that at each point $(x, y)$ on the curve, the tangent line has slope $x^2$. Find an equation for the curve given that it passes through the point $(2, 1)$.

### Solution:

★ Since the slope of the line tangent to $y = f(x)$ is $dy/dx$, we have

$dy/dx = x^2$, and $y = \int x^2\,dx = \frac{1}{3}x^3 + C$

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Since the curve passes through $(2,1)$, a specific value for can be found by using the fact that $y = 1$ if $x = 2$. Substituting these values in the above equation yields.

$$1 = \frac{1}{3}(2)^3 + C \quad or \quad C = -\frac{5}{3}$$

★ *So an equation of the curve is*

$$\boxed{y = \frac{1}{3}x^3 - \frac{5}{3}}$$

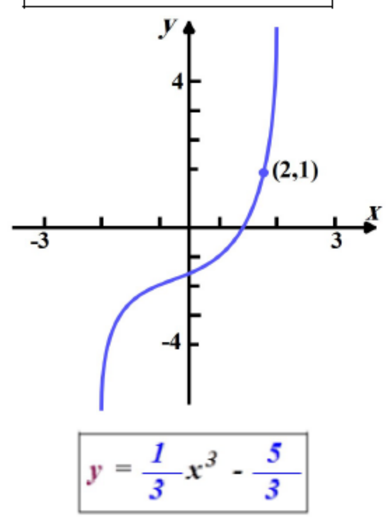


*Figure 5.2.2*

**Example (6):**

★ *Solve the initial-value problem*

$$\frac{dy}{dx} = \cos x \quad , \quad y(0) = 1.$$

## Solution:

★ The solution of the differential equation $dy/dx = \cos x$ is

$$y = \int \cos x \; dx = \sin x + C \qquad (11)$$

★ The initial condition $y(0) = 1$ implies that $y = 1$ if $x = 0$; substituting these values in (11) yields.

$1 = \sin(0) + C$ or $C = 1$

★ Thus, the solution of the initial-value problem is

$$\boxed{y = \sin x + 1}$$

| Derivative Formula | Equivalent Integration Formula |
|---|---|
| $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1}(x) + c$ |
| $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$ | $\int \frac{1}{1+x^2} \cdot dx = \tan^{-1}(x) + c$ |
| $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$ | $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}(x) + c$ |

# Homework

$$\int \frac{x^5 + 2x^2 - 1}{x^4} dx$$

$$\int \left[ \frac{2}{x} + 3\, e^x \right] dx$$

$$\int \frac{\sin x}{\cos^2 x} dx$$

$$\int \frac{\sec x + \cos x}{2\cos x} dx$$

$$\int \left[\frac{1}{2\sqrt{1 - x^2}} - \frac{3}{1 + x^2}\right] dx$$

**Solve the initial-value problems.**

(a) $\frac{dy}{dx} = \sqrt[3]{x}$ , $y(1) = 2$

## You Should Try

$$\int x\left(1+x^{3}\right)dx$$

$$\int\left[3\sin x - 2\sec^{2} x\right]dx$$

$$\int \sec x\left(\sec x + \tan x\right)dx$$

**Solve the initial-value problems.**

$$\frac{dy}{dt} = \sin t + 1 \quad , \quad y\left(\frac{\pi}{3}\right) = \frac{1}{2}$$