5.3 INTEGRATION BY SUBSTITUTION

$$\int f(u)\,du = F(u) + C$$

5.3 INTEGRATION BY SUBSTITUTION:

u-Substitution:

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

★ For our purpose it will be useful to let $u = g(x)$ and to write $du / dx = g'(x)$ in the differential form $du = g'(x)\, dx$.

With this notation (2) can be expressed as

$$\int f(u)\, du = F(u) + C$$

Example (1):

Evaluate:

$$\int (x^2 + 1)^{50} \cdot 2x \; dx.$$

Solution:

$$\int (x^2 + 1)^{50} \cdot 2x \; dx.$$

★ *If we let* $u = x^2 + 1$ *, then*

$$dx / dx = 2x ,$$

★ *Which implies that* $du = 2x\, dx$.

Thus , the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x \; dx = \int u^{50} \; du$$

$$= \frac{u^{51}}{51} + C = \frac{(x^2 + 1)^{51}}{51} + C$$

Guidelines for u-Substitution :

Step (1). Look for some composition within the integrand for which the substitution

$$\boxed{u = g(x)} \quad , \quad du = g'(x)\,dx$$

produces an integral that is expressed entirely in terms of u and its differential du . This may or may not be possible.

Step (2). If you are successful in Step(1), then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step (3). If we are successful in Step (2), then replace u by $g(x)$ to express your final answer in terms of x.

Example (2):

Evaluate:

(a) $\int \sin(x+9)\,dx$

(b) $\int (x-8)^{23}\,dx$

Solution:

(a) $\int \sin(x+9)\,dx$

★ If we let $\boxed{u = x+9}$,then

$$du = dx$$

$$\int \sin(x+9)\,dx = \int \sin u\,du$$

★ Remember that :

$$★ \int \sin u\,du = -\cos u + C$$

$$= -\cos u + C = \boxed{-\cos(x+9) + C}$$

(b) $\int (x-8)^{23}\ dx$

★ *If we let* $\boxed{u = x - 8}$ *, then*

$$du = dx$$

- **Thus, the given integral can be written as**

$$\int (x-8)^{23}\ dx = \int u^{23}\ du$$

$$= \frac{u^{24}}{24} + C = \boxed{\frac{(x-8)^{24}}{24} + C}$$

Example (3):

Evaluate:

$\int \cos\ 5x\ dx$.

Solution:

$$\int \cos\ 5x\ dx$$

★ *If we let* $\boxed{u = 5x}$ *, then*

$$du = 5\ dx\ ,$$

- **Which implies that** $\frac{1}{5}du = dx$.

- **Thus, the given integral can be written as**

$$\int \cos\ 5x\ dx = \frac{1}{5}\int \cos\ u\ du$$

$$= \frac{1}{5}\sin u + C = \boxed{\frac{1}{5}\sin\ 5x + C}$$

✍ Example (4):

Evaluate:

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} .$$

✎ **Solution:**

$$\int x^2 \, e^{-2x^3} \, dx$$

✎ **Solution:**

Remember:

Derivative Formula	Equivalent Integration Formula
$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1}(x) + c$
$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \cdot dx = \tan^{-1}(x) + c$
$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}(x) + c$

Example (5):

Evaluate:

$$\int \frac{dx}{1+3x^2}$$

Solution:

$$\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+\left(\sqrt{3}\,x\right)^2}$$

★ **Substituting** $\boxed{u = \sqrt{3}\,x}$ **, then**

$$du = \sqrt{3}\,dx \quad . \text{ So } \quad \frac{1}{\sqrt{3}} du = dx$$

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2} du$$

★ **Remember that :**

$$\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1} u + C = \boxed{\frac{1}{\sqrt{3}}\tan^{-1} \sqrt{3}\,x + C}$$

Example (6):

Evaluate:

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right)dx \ .$$

Solution:

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right)dx = \int \frac{1}{x}dx + \int \sec^2 \pi x\, dx$$

$$= \mathrm{Ln}\,|\,x\,| + \int \sec^2 \pi x\, dx$$

✸ To evaluate $\int \sec^2 \pi x\, dx$, substituting

$\boxed{u = \pi x}$, then

$du = \pi\, dx$. So $\frac{1}{\pi} du = dx$

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right)dx = Ln|x| + \frac{1}{\pi}\int \sec^2 u\, du$$

★ **Remember that :**

★ $\int \sec^2 u\, du = \tan u + C$

$$\boxed{= Ln\,|x| + \frac{1}{\pi}\tan \pi x + C}$$

Example (7):

Evaluate:

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$$

Solution:

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx = \int \frac{e^x}{\sqrt{1 - \left(e^x\right)^2}}\, dx$$

★ *If we let* $\boxed{u = e^x}$. *then*

$$du = e^x dx$$

Thus ,

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx = \int \frac{1}{\sqrt{1 - u^2}}\, du$$

★ ***Remember that :***

$$\star \int \frac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1} u + C$$

$$= \sin^{-1} u + C \quad \boxed{\sin^{-1} e^x + C}$$

Example (8):

Evaluate:

$$\int \cos^4 3t \sin 3t\, dx$$

Solution:

The generalizations of Formula:

$$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1} \frac{u}{a} + C , |u| < |a|$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C , |u| > |a|$$

✍ ***Example (9):***

Evaluate:

$$\int \frac{dx}{\sqrt{2 - x^2}}$$

✏ **Solution:**

Example (10):

Evaluate:

$$\int \frac{\sin\theta}{\cos^2\theta + 1}\, d\theta$$

Solution:

Less Apparent Substitutions:

- The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f \;\; g \;\; x$ and the remainder of the integrand is a constant multiple of $g' \;\; x$. If this is not the case, the method may still apply but may require more computation.

Example (11):

Evaluate:

$$\int x^2 \sqrt{x - 1} \;\; dx$$

Solution:

$$\int x^2 \sqrt{x - 1} \;\; dx$$

★ *If we let* $\boxed{u = x - 1}$. *then* $u + 1 = x$ **and**

$$du = dx$$

Thus ,

$$\int x^2 \sqrt{x - 1} \;\; dx = \int \;\; u + 1 \;\; ^2 \sqrt{u} \;\; du$$

$$= \int \;\; u^2 + 2u + 1 \;\; u^{1/2} \; du$$

$$= \int \;\; u^{5/2} + 2u^{3/2} + u^{1/2} \quad du$$

$$= \frac{u^{7/2}}{7/2} + 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{7}\ x-1\ ^{7/2} + \frac{4}{5}\ x-1\ ^{5/2} + \frac{2}{3}\ x-1\ ^{3/2} + C$$

Example (12):

Evaluate:

$$\int \cos^3 x\ dx$$

Solution:

$$\int \cos^3 x\ dx = \int \cos^2 x\ \cos x\ dx$$

★ **Remember that :**

$$\star\ \cos^2 x + \sin^2 x = 1$$

$$= \int\ 1 - \sin^2 x\ \ \cos x\ dx$$

* *We can then evaluate the integral by substituting*

$$\boxed{u = \sin x}$$

$$du = \cos x\ dx$$

* *Therefore,*

$$\int \cos^3 x\ dx = \int\ 1 - u^2\ \ du$$

$$= u - \frac{u^3}{3} + C$$

$$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$$

Example (13):

Evaluate:

$$\int \sin^3 x\ dx$$

(You Should Try)

Example (14):

Evaluate:

$$\int \sin^2 x \ \cos x \ dx \ .$$

Solution:

$$\int \sin^2 x \cos x \ dx = \int \sin^2 \cos x \ dx$$

✸ *Substituting*

$\boxed{u = \sin x}$, then

$$du = \cos x \ dx$$

Thus,

$$\int \sin^2 x \ \cos x \ dx = \int u^2 \ du$$

$$= \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$$

Example (15):

Evaluate:

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

Solution:

★ *If we let* $\boxed{u = \sqrt{x}}$ .then

$$du = \frac{1}{2\sqrt{x}}dx \ . \quad So \quad 2du = \frac{1}{\sqrt{x}}dx$$

Thus ,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{u}\,du$$

$$= 2e^{u} + C \quad \boxed{2e^{\sqrt{x}} + C}$$